Calculation of Free Turbulent Mixing by the Interaction Approach

THOMAS MOREL* AND T. PAUL TORDA†
Illinois Institute of Technology, Chicago, Ill.

The applicability of Bradshaw's interaction hypothesis to two-dimensional free shear flows was investigated. According to it, flow with velocity extrema may be considered to consist of several interacting layers. The hypothesis leads to a new expression for the shear stress which removes the usual restriction that shear stress vanishes at the velocity extremum. The approach is based on kinetic energy and the length scale equations. The compressible flow equations are simplified by restriction to low Mach numbers, and the range of their applicability is discussed. The empirical functions of the turbulence model are found here to be correlated with the spreading rate of the shear layer. The analysis demonstrates that the interaction hypothesis is a workable concept.

Nomenclature

a_1, L, G = see Eqs. (10)
e = $(\overline{u_i'u_i'} + \overline{\rho' u_i' u_i'}/\rho)/2$
k = $\overline{u_i'u_i'}/2$
L_0 = width to the half-velocity point
L_e, L_τ, l_k = length scales, see Eqs. (7), (8) and (14)
l_M = mixing length
P = $-\overline{u'v'}\,\partial U/\partial y$
Q = volume of turbulent fluid
R = U_2/U_1
U_j = jet nozzle velocity
ΔU = maximum velocity difference
$\tilde{V}$ = $V + \overline{\rho' v'}/\rho$
W = $|\Delta U|/U_1$, also see Eq. (28)
δ = see Eq. (11)
ε = dissipation
θ = $U_1^{-2}\int_0^\infty U(U-U_1)\,dy$, momentum thickness
ν_t = effective viscosity
τ = $-(\overline{u'v'} + \overline{\rho' u' v'}/\rho)$
τ^+, τ^- = shear stress in a "simple" layer

Subscripts

c = centerline
$e, 1, 2$ = edge of the flow
m, max = maximum value
1/2 = half-velocity point

I. Introduction

THE majority of the flows occurring in nature, which are of an interest in engineering, are turbulent. These flows are inherently unsteady, three-dimensional, and very complex. Because of that the solution of the complete equations of motion requires a very large number of calculation points, even if the time-averaged motion is two-dimensional and steady. To avoid this problem one may use the fact that it is the time average of the flow variables which is often the most important information sought. The restriction to the average conditions reduces the number of required calculation points to a level which can be handled by today's computers. The price for this simplification is very high as the averaged equations always turn out to be less numerous than the unknowns they introduce. The ensuing closure problem may be overcome only by relying heavily on experimental results.

The turbulent flows of greatest practical interest are thin shear layers of which only the group of free flows will be considered here: mixing layers, jets, and wakes. A wide range of approaches to the closure problem for these flows has been developed, ranging from the simplest eddy viscosity models to methods proposing the solution of the transport equations for the Reynolds stress tensor; the latest trends in prediction methods are summarized in Ref. 1.

The primary objective of the present work was to investigate the applicability of the interaction hypothesis of Bradshaw to two-dimensional free flows. According to the hypothesis, turbulent thin shear layer flows with velocity extrema may be considered to consists of several "simple" shear layers loosely divided by the velocity extrema. This has the crucial consequence that the shear stress is not required to vanish at the velocity extremum and allows use of the kinetic energy equation for predictions of asymmetric flows, which was previously considered not possible. The present approach is based on the turbulent energy equation modeled in the manner proposed by Bradshaw et al.[2] for boundary-layer calculations. Their analysis is extended here to free shear flows and to include the transport equation for the length scale. In the spirit of the interaction hypothesis, free flows with velocity extrema (jets and wakes) are computed as two neighboring mixing layers. The interaction hypothesis offers a meaningful alternative to other methods which are based on the kinetic energy equation, and which rely on the eddy viscosity near the velocity extremum, [e.g. Lee and Harsha[3] (implicitly) and Rodi and Spalding[4] (explicitly)]. Bradshaw et al.[5] applied the hypothesis to duct flows and showed its usefulness for studies of interacting wall flows. Here, the hypothesis was used for predictions of jets and wakes, and so the next logical step would be the application of the method to wall jets which combine the features of both the wall and free flows. The flows considered are two-dimensional, at high Reynolds numbers and at moderate Mach numbers. Some preliminary results of this work were included in Ref. 1 as the paper no. 16.

Presented as Paper 73-649 at the AIAA 6th Fluid and Plasma Dynamics Conference, Palm Springs, Calif., July 16–18, 1973; submitted July 30, 1973; revision received October 30, 1973. This research was supported jointly by a NASA Fellowship, by NASA Grant NGR-14-004-028, and by Themis Project Contract AF F44620-69-C-0022. Part of this work was done at the Imperial College of Science and Technology, London. The many valuable suggestions of P. Bradshaw are gratefully acknowledged.

Index category: Jets, Wakes, and Viscid-Inviscid Flow Interactions.

* Graduate Research Assistant, Mechanics, Mechanical and Aerospace Engineering Department; presently at Fluid Dynamics Research Department, GM Research Laboratories, Warren, Michigan. Member AIAA.

† Professor, Mechanics, Mechanical, and Aerospace Engineering Department. Associate Fellow AIAA.

II. Governing Equations

The flows considered are subject to the boundary-layer approximations. Further, the molecular diffusion terms, negligible at high Reynolds numbers, may be dropped from all equations. The average continuity and momentum equations

$$\partial U/\partial x + \partial V/\partial y = 0 \qquad (1)$$

$$U(\partial U/\partial x) + V(\partial U/\partial y) = -(1/\rho)(dp/dx) - \partial\overline{u'v'}/\partial y \qquad (2)$$

have three unknowns and, therefore, an additional equation is required to close the problem. The simplest method of closure is based on the fact that $\overline{u'v'}$ appears in Eq. (2) in the same form as the molecular shear stress. This suggested to many that $\overline{u'v'}$ could be related to $\partial U/\partial y$ by an "effective viscosity" v_t or through a "mixing length" l_M as

$$-\overline{u'v'} = v_t(\partial U/\partial y) \qquad \text{or} \qquad -\overline{u'v'} = l_M{}^2|\partial U/\partial y|\partial U/\partial y \qquad (3)$$

Much effort was devoted to establish the suitable length and velocity scales of the mean flow needed to render v_t and l_M dimensionless and to find the constants of proportionality. This approach actually gives acceptable results in selfpreserving flows where the scales of turbulence and of the mean flow are proportional. On the other hand, it is clear that $\overline{u'v'}$ must be related to the local turbulence level.

To gain information about turbulence one may turn to higher order transport equations. Multiplying the Navier-Stokes equations by the velocity vector six new equations are obtained which, upon time averaging, become independent of the mean momentum equation and carry new information. These are the transport equations for Reynolds stresses. This set of equations is not closed as it contains unknown higher order correlations. To eliminate them it is usually assumed that they may be expressed in terms of other quantities such as $\partial U_i/\partial x_j$ and $\overline{u_i'u_j'}$ (such relations will be referred to as structure). The modeling of the Reynolds equations is not an easy task and, therefore, simpler methods are often used utilizing only some of these equations. Two of them are of immediate interest. The first one

$$\frac{D}{Dt}(-\overline{u'v'}) = \overline{v'^2}\frac{\partial U}{\partial y} - \overline{\frac{p'}{\rho}\left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}\right)} + \frac{\partial}{\partial y}\overline{\left(v'^2 + \frac{p'}{\rho}\right)u'} + 2v\overline{\frac{\partial u'}{\partial x_k}\frac{\partial v'}{\partial x_k}} \qquad (4)$$

is attractive because it gives $\overline{u'v'}$ itself. However, the modeling of the important pressure rate-of-strain correlation, the second term on the right-hand side of Eq. (4), is still a major problem. By contrast, the equation for $k = \overline{u_i'u_i'}/2$ is relatively well documented experimentally. It may be written as

$$\frac{Dk}{Dt} = -\overline{u'v'}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\overline{p'v'}}{\rho} + \frac{1}{2}\overline{u_i'u_i'v'}\right) - \varepsilon \qquad (5)$$

where the viscous terms have been approximated, for high Reynolds numbers, by the dissipation ε. Equation (5) does not contain the pressure rate-of-strain correlation, which vanishes in contraction. The pressure-velocity correlation appearing in Eq. (5) is not as important, and is generally treated together with the triple velocity correlation as a lateral diffusion because the two terms appear in a similar form. The terms of Eq. (5) are referred to, from left to right, as convection, production, diffusion, and dissipation.

III. Modeling of the Turbulent Energy Equation

There are two currently used approaches to modeling of the shear stress: 1) within the fully turbulent region (except in the vicinity of a velocity extremum) there is a good amount of evidence to support

$$\tau = 2a_1 k \qquad (6)$$

where a_1 is approximately constant[6] having values varying within a small range depending on the type of flow considered. Near a velocity extremum and at the edge of the turbulent region a_1 decreases to zero; and 2) the eddy viscosity concept[7]

$$\tau = k^{1/2}L_\tau(\partial U/\partial y) \qquad (7)$$

Here, the velocity scale defining the eddy viscosity is determined by the turbulence; however, the shear stress is still assumed proportional to the mean velocity gradient. This formulation was applied to free turbulent flows by Rodi and Spalding.[4] Lee and Harsha[3] chose the formulation (6), except that near centerline of jets and wakes they used, in effect, a hybrid of Eqs. (6) and (3).

There is a general agreement on the treatment of the dissipation which, in high Reynolds number flows, is modeled as

$$\varepsilon = k^{3/2}/L_\varepsilon \qquad (8)$$

The diffusion term is subject to some controversy but it is agreed it may be viewed as consisting of two contributions—a gradient diffusion and a bulk convection

$$\overline{p'v'}/\rho + \tfrac{1}{2}\overline{u_i'u_i'v'} = -C_k(\partial k/\partial y) + V_k k \qquad (9)$$

where C_k is the eddy diffusivity and V_k is a diffusive velocity representing the convection by large scale motions. The controversy is over the relative importance of these two contributions. Expressions similar to Eq. (9) may be written for the transport of other quantities, and experiments suggest that the relative importance of the two contributions depends on the property transported.[6] The usual approach in prediction methods is to consider, for simplicity, only one of the two contributions and to neglect the other one.

The modeling of the turbulent energy equation employed in this work is that originally proposed by Bradshaw et al.[2] The physics behind the assumptions of the model is well confirmed by experiments and by the success of the method itself. The structure of turbulence is defined by Bradshaw et al., to be of the following form:

$$\tau = 2a_1 k \qquad (10a)$$

$$\varepsilon = \tau|\tau|^{1/2}/L \qquad (10b)$$

$$\overline{p'v'}/\rho + \tfrac{1}{2}\overline{u_i'u_i'v'} = V_\tau \tau = G|\tau_m|^{1/2}\tau \qquad (10c)$$

Equations (10) define the empirical functions a_1, L and G. The numerical values of these functions, as obtained from the free shear-flow data, are found to be different from those applicable to boundary layers (see Sec. V). The local velocity and length scales of turbulence are assumed to be proportional, respectively, to $|\tau_{\max}|^{1/2}$ and to δ defined by

$$\delta = 1.96\int_0^\infty \frac{\tau}{\tau_m}dy \qquad (11)$$

This choice makes the definitions (10) independent of the mean velocity profile and, therefore, consistent from the point of view of the interaction hypothesis discussed in Sec. IV. The diffusion is represented by the bulk convection Eq. (10c). This is based on the argument that the free shear flows exhibit strong large scale motions which tend to make the bulk diffusion very important. The apparent bulk transport velocity V_k, which is at its maximum near the edges of the flow, reaches values on the order of $0.1\Delta U$. The dissipation length L is typically on the order of 0.1δ.

A common approach in other prediction methods which use the kinetic energy equation is to solve Eq. (5) for k and then to obtain τ either from Eq. (6) or (7). In contrast to that, Bradshaw et al., eliminate k by Eq. (10a) from Eq. (5) and thus convert it into an empirical transport equation for τ.

$$\frac{D}{Dt}\frac{\tau}{2a_1} = \tau\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}(G\tau|\tau_{\max}|^{1/2}) - \frac{\tau|\tau|^{1/2}}{L} \qquad (12)$$

Remarks on the Shear Stress Equation

Despite the difficulties with the pressure rate-of-strain term (PRS) several workers have proposed models for the shear stress Eq. (4). Some of these models are compared below to Eq. (12). Of the terms of Eq. (4) the first two, convection and generation, may be measured. The last term may probably be neglected, because at high Reynolds numbers the small scale fluctuations (which contribute to it) are nearly isotropic. Cross-stream integration eliminates the third term on the right-hand side, the diffusion. Now, since convection is usually much smaller than generation, the remaining term, PRS, must be of the same order of magnitude as generation. It follows that *PRS* is very large and, therefore, crucial to the success of predictions. It tends to reduce the magnitude of τ and is responsible for the tendency of turbulence towards isotropy. The early model of Rotta[8]

$$PRS_{ij} \sim (\overline{u_i'u_j'}/2k - \tfrac{1}{3}\delta_{ij})\varepsilon \qquad (13)$$

reflects this fact and postulates that the tendency to isotropy is proportional to the departure from isotropy. Donaldson[9] adopted this model of *PRS* (and in addition assumed that the viscous term is also proportional to the dissipation).

The governing Poisson's equation for pressure fluctuations suggests a different model because one of the two forcing terms in this equation is dependent on the mean rate of strain. The general solution of this equation is known and may be used to express *PRS* in terms of volume and surface integrals.[10] The integrands of these integrals are two-point correlations whose values decay sharply with separation. This suggests that, despite the global dependence, one may attempt to model *PRS* locally. The existence of the forcing term involving the mean rate of strain in the pressure equation prompted development of *PRS* models having two parts. The first part is usually modeled in a manner similar to Rotta's Eq. (13) and the second part is expressed as a product of the double velocity correlations and mean rate of strain. This approach was followed by Rotta[11] and by Hanjalić and Launder.[12] Rotta chose to model the second part of *PRS* together with the production term, while Hanjalić and Launder model *PRS* alone.

The shear stress equations of Donaldson, Rotta,[11] and Hanjalić and Launder may be compared with the equation proposed by Bradshaw et al.

$$(D\tau/Dt)_D = a_{v\tau}P - 10a_1\varepsilon + \text{Diffusion}$$

$$(D\tau/Dt)_R = (0.045/a_1)P - 2a_1\varepsilon + \text{Diffusion}$$

$$(D\tau/Dt)_{HL} = (0.098/a_1)P - 5.6a_1\varepsilon + \text{Diffusion}$$

$$(D\tau/Dt)_B = 2a_1P - 2a_1\varepsilon + \text{Diffusion}$$

where

$$a_{v\tau} = \overline{v'^2}/-\overline{u'v'}, \quad P = \overline{u'v'}\,\partial U/\partial y$$

and the subscripts refer to the authors. The last equation (B) is to be applied only to "simple layers" in the sense of Sec. IV, boundary layers, and mixing layers, while the other equations can treat flows with shear stress reversal directly.

All above equations are essentially alike. The differences in the coefficients of production and dissipation terms come mostly from the modeling of *PRS*. The disagreement on modeling of *PRS* indicates the need of continued search for its suitable representation. (Note, however, that there is no guarantee that one can find a universal structure valid for all turbulent flows). Therefore, in the meantime, it appears worthwhile to develop less complex models (such as the shear stress equation of Bradshaw et al.) utilizing available experimental evidence for limited classes of flows.

Equation for the Length Scale of Turbulence

The length scale L is found here to be adequately represented as being a constant across the layer and so one has to determine only its magnitude and development. The first approach to this problem was to correlate L/δ with the local spreading rate $d\delta/dx$ (Sec. V) which will be referred to as Version *A*. An alternate approach is to use a rate equation for L. To arrive at such an equation, one may start from the exact equation for kl_k derived by Wolfstein[13] by manipulation of the dynamic equation for the two-point velocity correlation $\overline{u'_{i,A}u'_{j,B}}$

$$(D/Dt)kl_k = l_G P + \text{Diffusion—Viscous Terms} \qquad (14)$$

where

$$l_k = \frac{1}{k}\int \overline{u'_{i,A}u'_{j,B}}\,\frac{dV}{r^2}$$

If all the length scales (L, l_G, l_k) are assumed to be proportional to each other, then Eq. (14) may be rewritten as an equation for kL. Further, in the absence of any information about the viscous term of the kL equation one may assume it proportional to ε. Then, using Eq. (5) k may be eliminated from the kL equation to give an equation for L itself

$$DL/Dt = (L/k)(C_1'P + C_2'\varepsilon) + \text{Diffusion} \qquad (15)$$

The diffusion term of Eq. (15) may be modeled to be either of the bulk type or as a gradient diffusion but in the present context the details of diffusion are not too important as it is sufficient to evaluate Eq. (15) only at one characteristic position. Therefore, as the first approximation, the contribution of the diffusion term at the characteristic position was absorbed into the production and destruction terms

$$U(dL/dx) = L/2k(C_1P + C_2\varepsilon) \qquad (16)$$

Eq. (16) with the half-velocity point as the characteristic position, was applied to mixing layers, jets, and wakes. In all cases adequate results were obtained by using $C_1 = 0$. The constant C_2 equals 0.57 for jets and wakes, while for mixing layers $C_2 = 5a_1$. The rate equation for L will hence be referred to as Version *B*.

IV. Flows with Velocity Extrema

In asymmetric flows with velocity extrema, such as wall jets, ducts with walls of unequal roughness or annular flow, the positions of $\tau = 0$ and $\partial U/\partial y = 0$ do not coincide in general. That poses a problem for methods which do not use the shear stress Eq. (4), because they cannot account for the diffusion and convection of τ over the velocity extremum. These methods usually depend on one of the shear stress models Eqs. (3), (6) or (7). Therefore, their direct application is not possible as that would require the a priori knowledge of the position of the point where $\tau = 0$ and would lead to physically meaningless values of v_t, l_M, and L_τ.

The velocity extremum (in the two-dimensional case) separates the flow into two regions each of them having its own large scale motions which dominate it. Within these regions the shear stress tends to have the same sign as the mean velocity gradient. The two large scale motions are not sharply divided by the extremum but they cross it alternately carrying with them fluid with shear stress of a sign corresponding to the region of origin. This shear stress (the short time average) will not change sign immediately upon crossing the centerline. An intermittency γ may be considered measuring the fraction of time during which the fluid passing through a point is coming from one side of the stream only (say top or bottom). Then one may define conditional averages of the shear stress

$$\tau^+ = \gamma\tau_{\text{top}} \qquad \tau^- = (1-\gamma)\tau_{\text{bottom}}$$

It may be argued that it is only the conventional average over the alternate contributions

$$\tau = \tau^+ + \tau^- \qquad (17)$$

which makes a_1 go through zero near a velocity extremum. In symmetric flows τ will be zero on the centerline (Fig. 1).

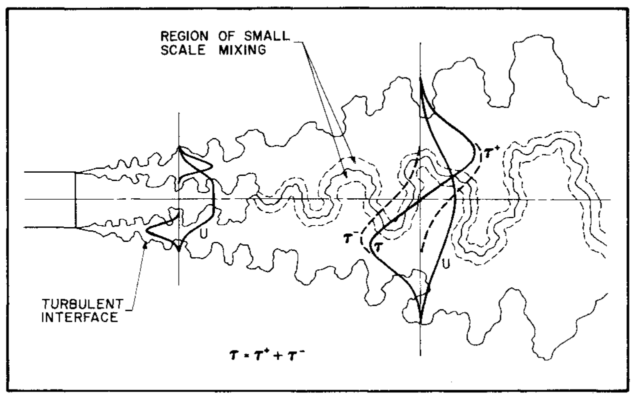

Fig. 1 Shear layer interaction.

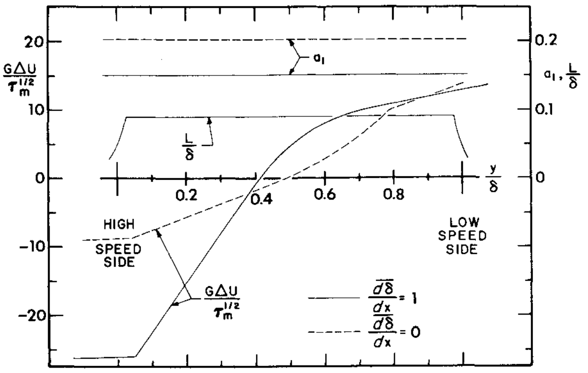

Fig. 2 Empirical functions—mixing layers.

What has been described above is a qualitative picture, supported by experimental evidence, depicting flows with velocity extrema as consisting of more or less distinct regions. As a consequence one may consider Eq. (17) to be a meaningful alternative to Eqs. (3), (6), and (7). The advantage of Eq. (17) is that it does not require $a_1 = 0$ at $\tau = 0$, nor does it relate the shear stress to $\partial U/\partial y$.

Interaction Hypothesis

The interaction hypothesis was formulated by Bradshaw et al.[5] They proposed that a flow with a velocity extremum may be regarded as two separate "simple" layers. Further, they suggested that, to a crude approximation, the two neighboring turbulent fields do not interact with each other. The turbulence in each region is shaped by the mean velocity profile (and, when applicable, by the vicinity of a wall). The only information the turbulence has about the existence of the other layer is through the mean velocity profile. In other words, the structure functions of the simple layers are to be only little or not at all affected by the interaction. The simple layer itself is a flow in which the shear stress does not change sign and which has one set of large eddies only. Depending on the boundary conditions it is either a "boundary layer" or a "mixing layer." If the hypothesis were exactly true then all complex thin shear layer flows would consist only of a combination of these two simple layers.

The hypothesis is attractive in that the empirical functions of each layer of a complex flow should not be very different from those in the corresponding simple layers. (In duct flows they were actually found by Bradshaw et al.[5] to be exactly the same as in the boundary layers; the degree of differences between the structures of jets and mixing layers may be seen in Figs. 2 and 3). Further it allows the use of the well-documented kinetic energy equation for calculations of wider class of flows than previously considered possible. In numerical application, all layers of the flow share the profiles of U and V but have their own, separate, shear stress profiles which overlap (Fig. 1). The shear stresses are obtained from separate equations

$$\begin{aligned} \frac{D}{Dt}\left(\frac{\tau^+}{2a_1}\right) &= \tau^+\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}(G|\tau_m{}^+|^{1/2}\tau^+) - \frac{\tau^+|\tau^+|^{1/2}}{L} \\ \frac{D}{Dt}\left(\frac{\tau^-}{2a_1}\right) &= \tau^-\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}(G|\tau_m{}^-|^{1/2}\tau^-) - \frac{\tau^-|\tau^-|^{1/2}}{L} \end{aligned} \quad (18)$$

The conventional shear stress, needed in the momentum equation, is obtained by addition as in Eq. (17).

$$\frac{DU}{Dt} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial \tau^+}{\partial y} + \frac{\partial \tau^-}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial \tau}{\partial y} \quad (19)$$

One may ask whether this approach can be extended to axisymmetric flows. We feel that numerically it is possible, but physically one really ought to be talking about an infinite number of interactions and not of just two layers. Incidentally, this argument also raises the question whether the difference between axisymmetric and 2-D flows does not amount to more than the formal inclusion of the radius into the lateral transport term $1/r\,\partial/\partial r$ ($r \times$ transport) while keeping the structure unchanged—which is the commonly used approach.

V. Empirical Functions

The empirical functions L and G may be obtained from the experimental kinetic energy balance and the shear stress profile. However, the measurements tend to be inaccurate near the edges. That creates problems for determination of G (L is less important in that region than G), but fortunately this problem may be circumvented. It may be shown that near the edges of free shear flows, similarly as at the edge of a boundary layer,[2] G may be determined from the entrainment rate by

$$dQ/dx = 2a_1 G|\tau_m|^{1/2} \quad (20)$$

It is seen that Eq. (20) is a condition on $a_1 G$ rather than on G alone. This gives the freedom to use constant a_1 across the layer as an acceptable approximation. Further, while $1/\Delta U\, dQ/dx$ changes from 0.055 in a freejet to 0.19 in a small-deficit wake, the product

$$(1/a_1)(\Delta U^2/\tau_m)(1/\Delta U)(dQ/dx)$$

(where a_1 is the value from the fully turbulent region) is fairly universal and, therefore, the extra factor $\Delta U/|\tau_m|^{1/2}$ was incorporated into the definition of G as in the case of the boundary layers.[2]

Mixing Layers

For the case of mixing with still air we have selected the experiments of Liepmann and Laufer[14] and of Bradshaw et al.[15] as most reliable. Good agreement between the calculations and the experiments was obtained using $a_1 = 0.15$ and $L/\delta = 0.09$. The position of $G = 0$ was found to be an important parameter influencing the shape of the velocity profile. Therefore, the position of $G = 0$ was fixed with respect to the shear stress maximum rather than with respect to the edge of the flow. For the two-stream mixing layers we have used the data of Spencer.[16] From comparison of the calculated velocity profiles with his results it was concluded that a_1 depends on the velocity ratio R. It appears to be increasing with R from 0.15 to $R = 0$ to 0.20 at $R = 1$. Experimental data on various mixing layers found in literature give no conclusive support to this finding as there is too much disagreement in the reported values of $\overline{u'v'}$ and of the fluctuating velocity components. L/δ appears unaffected by the velocity ratio, but G changes substantially.

The dependence of a_1 and G on the velocity ratio may also be correlated with the spreading rate $d\delta/dx$ which is a plausible physical quantity as one may write

$$(\partial U/\partial x)/(\partial U/\partial y) \sim (\Delta U/\Delta x)/(\Delta U/\Delta y) \sim d\delta/dx \quad (21)$$

It follows that $d\delta/dx$ is an integral quantity which may be considered to represent the rate-of-strain ratio $(\partial U/\partial x)/(\partial U/\partial y)$ within the layer. This ratio, in turn, was considered by

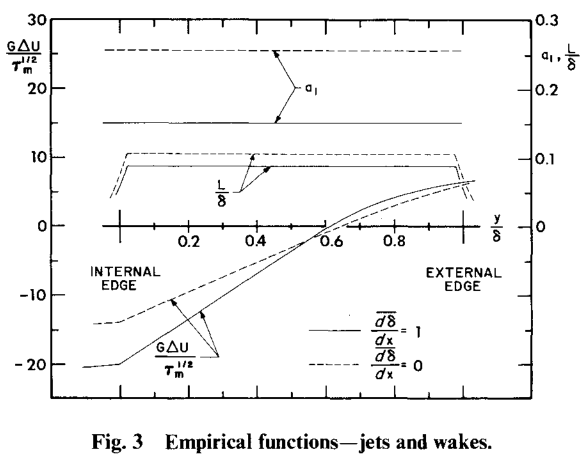

Fig. 3 Empirical functions—jets and wakes.

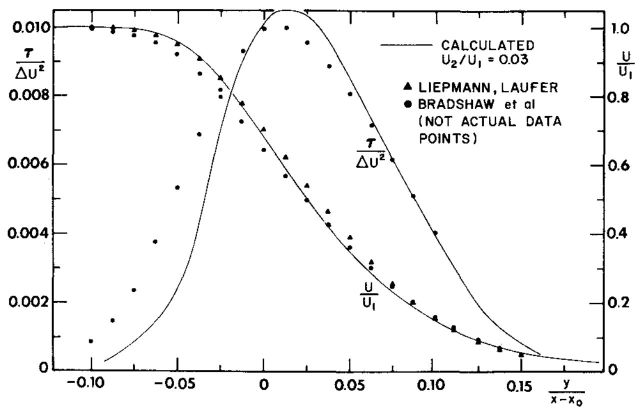

Fig. 4 Mixing layer with $R = 0$.

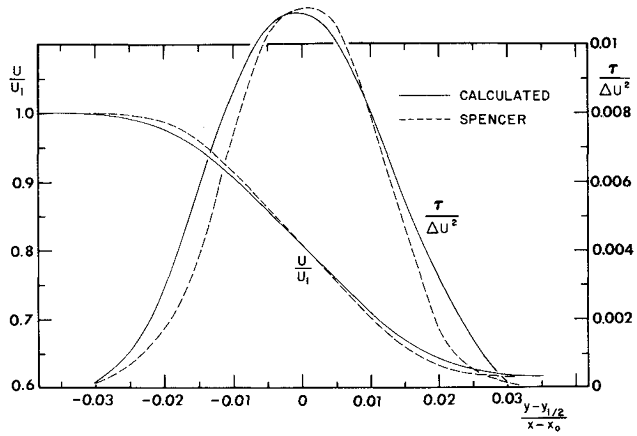

Fig. 5 Mixing layer with $R = 0.61$.

Townsend[6] to have an important influence on the development of large eddies in a turbulent flow. Therefore, a_1 and G were defined to be functions of the spreading rate. Their values for the limiting case of maximum and zero spreading rates are shown in Fig. 2. ($d\delta/dx$ is the local spreading rate of the simple layer, normalized by the spreading rate for $R = 0$).

Jets and Wakes

The empirical functions required for jets were found to be somewhat different from those used for mixing layers, which was interpreted as the effect of the interaction. The interaction modifies less the shape of these functions than their magnitude, which gives support to the interaction hypothesis. L and a_1 remain constant across the layer, with values $L/\delta = 0.087$ and $a_1 = 0.15$, but the effect on G is more profound (Figs. 2 and 3). The structure functions are not unique for all jets and wakes but they appear, again, to depend on the spreading rate. For a jet issuing into still air best agreement is found for $L/\delta = 0.087$ and $a_1 = 0.15$ while for a small-deficit wake $L/\delta = 0.106$ and $a_1 = 0.255$. The last number is supported by the results of Townsend[17,18] who reports a value of a_1 in excess of 0.2. The dependence on $d\delta/dx$ gave better results than correlations with the mean velocity ratio or with maximum shear stress, which were also tried, especially for the case with pressure gradient. The success of the $d\delta/dx$ correlation suggests that the spreading rate may really be a relevant parameter of the structure of turbulence. The dependence of the empirical functions on $d\delta/dx$ was approximated by a cosine curve; the two limiting forms are shown in Fig. 3.

In summary, the flows with velocity extrema may be regarded as weakly interacting quasi-parallel shear layers. There are more effects of the interaction on the empirical functions of free flows than in the case of ducts. The shear stress profiles of the "simple" layers of jets overlap more than those of a duct flow, and the value of the shear stress on the centerline of a jet or a wake is typically $0.5\tau_m$, while the corresponding value for a duct flow is only $0.1\tau_m$.[5] This fact is apparently responsible for the increased effects of the interaction on the empirical functions.

As for sensitivity of the calculations to a_1, L, and G—most important is L, which is about as influential as a mixing length would be. The other two functions are somewhat less important, but apparently have more influence here than in boundary layers. The exponential decay of L at the edges was incorporated for economy, to reduce the number of points in the "tail" of the profiles.

VI. Method of Solution

The modeling of diffusion in the kinetic energy equation has one important consequence. If the bulk convection model (10c) is used, the set of Eqs. (1, 2, and 12) becomes hyperbolic and can be solved either by a procedure suitable for parabolic equations or by the method of characteristics. The much simpler method of characteristics, successfully applied to the boundary layers,[2] was used here. The numerical method has difficulties in calculating flows issuing into small external streams. The angle of ingoing characteristics at the edge is equal arctan(V/U) and so it may reach exceedingly large values. When $U = 0$ the equations fail. However, the boundary-layer equations themselves are not valid for $V > U$, and other methods of solution would encounter problems, too. The difficulties near the edges could be overcome by the use of a special numerical treatment for the outer boundary. However, we did not find it necessary to implement such a treatment at the present time.

The sign of the empirical functions depends on the sign of τ and, therefore, on the coordinate system. To avoid having the functions change sign with the coordinate system we may use the fact that a_1, L, and G appear as products a_1G and a_1/L and so they may be kept always of the same sign. The only exception is the first term on the right-hand side of Eq. (12) which has to be changed to $|\tau|\,\partial U\,\partial y$. All calculations were performed on UNIVAC 1108 digital computer. Typical run time for the cases presented here was less than 20 sec.

VII. Incompressible Results

The predictions of Versions A and B are very close to each other for most of the calculated flows and, therefore, they are discussed separately only in the cases where more significant differences occur.

Mixing Layers

The case of mixing with still air is presented in Fig. 4. For numerical reasons the calculations were performed with

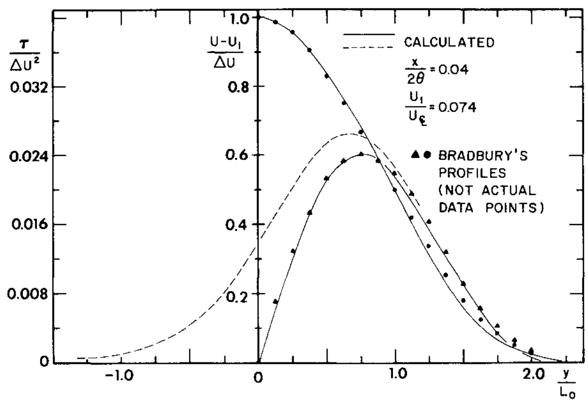

Fig. 6 Jet in a small external flow.

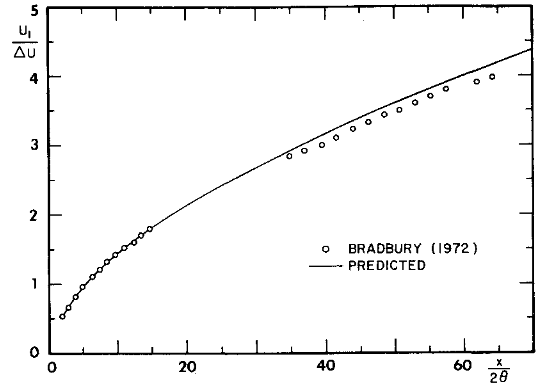

Fig. 7 Jet in a streaming flow.

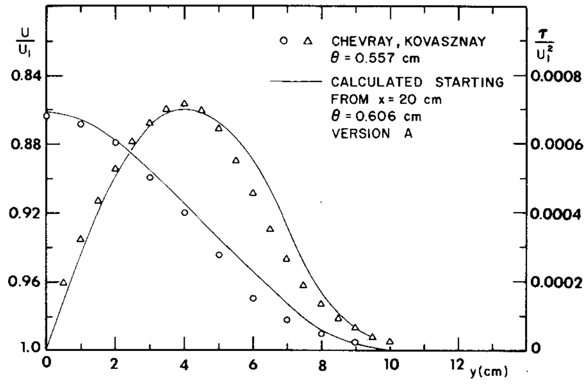

Fig. 9 Wake of a thin plate—profiles at $x = 240$ cm.

$R = 0.03$, where $R = U_2/U_1$, rather than zero, which reduced the spreading rate by about 6%. Two-stream mixing with $R = 0.61$ is compared to the data of Spencer[16] in Fig. 5. Spencer also obtained the spreading rates for six other velocity ratios between 0 and 0.76 and found them to follow well the relation

$$\sigma_0/\sigma = (1-R)/(1+R)$$

The calculated spreading rates lie very close to this curve, while $\tau_m/\Delta U^2$ varies with R within only a 10% range with a maximum around $R = 0.5$.

Jets and Wakes in Zero Pressure Gradient

A jet in a small external stream is compared with the data of Bradbury[19] in Fig. 6, where the broken line denotes the shear stress profile of the hypothetical simple layer. A jet in a moderate external stream (Bradbury[20]) is presented in Fig. 7. The calculations were continued up to very large values of $x/2\theta$ and they predict that the flow tends towards a self preserving state, which is the same as for wakes. Version B predicts a little faster approach to the final equilibrium than Version A. The mean velocity profiles are approximately similar throughout and

$$dL_0/dx = 0.1W \quad \text{at} \quad W \ll 1 \tag{22}$$

The calculated development of the spreading rate during the decay of a jet in a small external stream to a small-excess jet (at constant pressure) may be closely approximated by an empirical relation

$$dL_0/dx = [0.109W/(W+2)+0.1/(W+1)]W/(W+1) \tag{23}$$

Equation (23) may serve in engineering applications as a tool for quick estimates of the spreading rates of constant pressure jets. It is preferable to the expressions proposed by Bradbury[21] which, for small W, predict a spreading rate which is only 40% of that given by Eq. (22).

The prediction of a small-deficit wake is compared in Fig. 8 to the data of Townsend.[16,17] Far downstream both versions give the same result [with spreading rates following Eq. (22)], but Version B again tends faster towards the final equilibrium. Very interesting flow is the wake near the trailing edge of a thin plate.[22] Close to the trailing edge the structure of turbulence must be of the boundary-layer type rather than of the free flow type. However, already at $x = 20$ cm ($x/2\theta = 17.3$) the profiles of mean velocity and of turbulent quantities take on wake-like shapes and one may expect that the structure of turbulence also comes close enough to the wake type by that distance. To test this possibility the calculations were started at three different points: 0, 20, 50 cm using the experimental profiles as input. The last two calculations were found to agree with each other, but to differ from the first one, which tends to support the above expectation. Version B, with initial $L/\delta = 0.09$, gives similar results as Version A. In addition to that, it agrees well with the experiment when starting at $x = 0$ cm with $L/\delta = 0.065$. Figure 9 shows the profiles at the last measured station. Note that the experimental momentum thickness was 0.578 cm at separation, but equaled 0.606, 0.585, 0.557 cm at $x = 20$, 50, 240 cm, respectively.

Merging Region of a Two-Dimensional Jet

The interaction hypothesis allows calculation of the merging region without any change in the boundary conditions at the point where the mixing layers meet. However, the two layers overlap, act on each other and the structure of turbulence changes to the jet type. At the same time τ_c/τ_{max} increases from zero to about 0.5. This ratio was used as the parameter of the

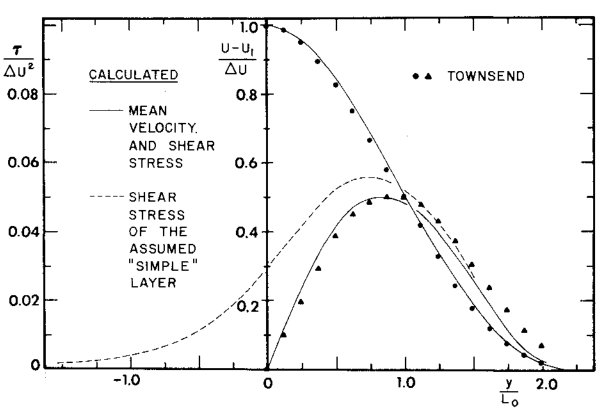

Fig. 8 Small-deficit wake.

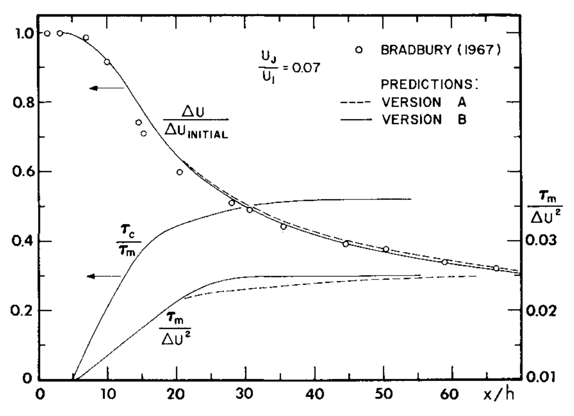

Fig. 10 Merging region of a two-dimensional jet.

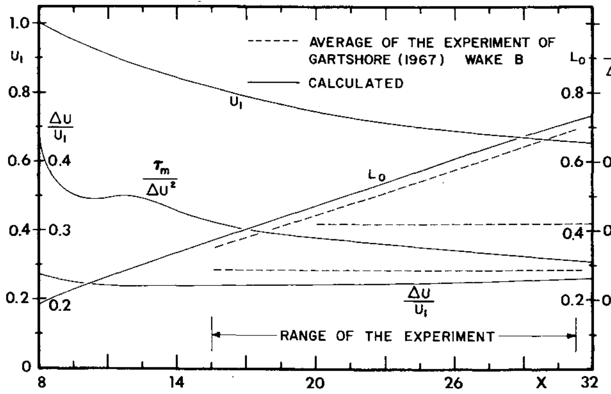

Fig. 11 Wake in a pressure gradient.

changing structure from a mixing layer to a jet, and a linear dependence of the structure on τ_c/τ_{max} was assumed (the predictions were quite insensitive to the form of this dependence). Figure 10 compares the calculations with the data of Bradbury[21] and shows that Version B indicates a faster increase in $\tau_m/\Delta U^2$ than Versions A.

Jets and Wakes in Pressure Gradients

When the external flow varies as x to the power $(-1/b)$, there are ranges of b for which the momentum equation allows selfpreservation of jets and wakes with

$$L_0 \sim x, \quad W = \text{constant and} \quad H = b-2 \tag{24}$$

Gartshore[23] measured two wakes in pressure gradients with b equal to 3.16 and 3.20. He found the wakes approximately selfpreserving, the latter one more so. We have calculated the latter wake and found that although the flow had the tendency to conform to the selfpreservation [Eqs. (24) were quite well satisfied] it was drifting away from it (Fig. 11). The data of Gartshore exhibit considerable scatter and only their mean is presented for comparison. The mean velocity profiles agree very well with Gartshore's. A jet in a decelerating stream with $b = 2.4$ was also calculated. The flow was predicted to be closely selfpreserving with $W = 1.43$, $L_0/x = 0.053$ and $x/2\theta = 3.13$. This result appears plausible, but there was no experiment available to us for comparison. The shear stress profiles of the simple layers in the last two flows overlap less than in zero pressure gradient and in both cases τ_c/τ_{max} is about 0.45.

VIII. Compressible Flow

The compressible flow equations are much more complicated than the incompressible ones as they contain a number of density-velocity correlations. If the restriction is made to consider only low and moderate Mach number flows, the equations may be substantially simplified. In the derivation of the simplified equations we have followed the work of Bradshaw and Ferriss[24] with small modifications. Their analysis was extended to include heterogeneous flows, too. Only the resulting equations are given below, but more detail may be found in Ref. 25. Based on the estimate of the order of magnitude of the density fluctuations, the requirement is made that

$$(\gamma-1)M^2(u'/U) \ll 1 \tag{25}$$

which allows dropping of several of the density-velocity correlations from the equations. As a result, the governing equations differ formally from the incompressible ones only by the inclusion of the effects of local mean density

$$\partial\rho U/\partial x + \partial\rho\tilde{V}/\partial y = 0$$

$$U\frac{\partial U}{\partial x} + \tilde{V}\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial\rho\tau}{\partial y}$$

$$U\frac{\partial \bar{e}}{\partial x} + \tilde{V}\frac{\partial \bar{e}}{\partial y} = \tau\frac{\partial U}{\partial y} - \frac{1}{\rho}\frac{\partial}{\partial y}\left[\rho \overline{v'\left(\frac{p'}{\rho} + e\right)}\right] - \varepsilon$$

where

$$\tilde{V} = V + \overline{\rho' v'}/\rho$$

$$\tau = -(\overline{u'v'} + \overline{\rho' u' v'}/\rho)$$

$$e = (u_i'u_i' + \rho' u_i' u_i'/\rho)/2$$

It remains to determine the structure of turbulence in compressible flows and the mean density variation across the flow. For the structure of turbulence one can rely on the suggestion of Morkovin[26] that the turbulence is convected passively by the mean flow as long as the local Mach number of the rms. fluctuations

$$\left(\frac{\rho \overline{u'^2}}{\gamma p}\right)^{1/2} = \frac{(\overline{u'^2})^{1/2}}{U}M \ll 1 \tag{26}$$

is small. The inference is that the structure is then Mach number independent and the same empirical functions may be used as for incompressible flow. Therefore, the structure definitions are analogous to those made in incompressible flow

$$\tau = 2a_1\bar{e}$$

$$\varepsilon = \tau|\tau|^{1/2}/L$$

$$\overline{v'\left(\frac{p'}{\rho} + e\right)} = G\tau|\tau_m|^{1/2}$$

In the Mach number range to which this approach is limited, a good approximation for the density profile can be obtained from the Crocco formula

$$c_p T + 0.5rU^2 = \text{constant} \tag{27}$$

where r is the recovery factor. Equation (27) together with the equation of state yields a relation $\rho = \rho(U)$ which eliminates the need for modeling of an additional partial differential equation. The final form of the equations is

$$\frac{\partial U}{\partial x} + \frac{\partial \tilde{V}}{\partial y} = -W\left(U\frac{\partial U}{\partial x} + \tilde{V}\frac{\partial U}{\partial y}\right) - \frac{U}{p}\frac{dp}{dx}$$

$$U\frac{\partial U}{\partial x} + \tilde{V}\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial\tau}{\partial y} + W\tau\frac{\partial U}{\partial y}$$

$$\left(U\frac{\partial}{\partial x} + \tilde{V}\frac{\partial}{\partial y}\right)\frac{\tau}{2a_1} = (1 - WG|\tau_m|^{1/2})\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}(G\tau|\tau_m|^{1/2}) - \frac{\tau^{3/2}}{L}$$

where

$$W = r(\gamma-1)M^2/U \tag{28}$$

Results

The effect of compressibility on the spreading rate of mixing layers was a subject of numerous experiments. Their consensus is that the effect of Mach number is to reduce the spreading rate, but they disagree on the magnitude of this reduction (see Paper No. 2 of Ref. 1). The calculations predict correctly the trend of the Mach number effect, but show it to be less than indicated by most of the experiments (see Paper No. 16 of Ref. 1). This is probably due to the violation of the assumptions used to simplify the equations, in particular the assumption $u'/U \ll 1$ made in Eqs. (25) and (26). This assumption is not justified in flows which issue into a small (or zero) external stream. In the absence of a wall the level of turbulence in jets and mixing layers is much higher than in boundary layers, especially at higher Mach numbers. Consequently, some of the terms dropped from the equations may become important when $M > 1$. Further, the structure of turbulence, which was assumed to be unaffected by compressibility, may begin to change. On the other hand, flows with large external streams satisfy $u'/U \ll 1$ well and the analysis should be applicable to wakes and small-excess jets up to moderate Mach numbers. The successful prediction of the supersonic wake at $M = 2.9$, measured by Demetriades,[27] tends to support this conclusion (Fig. 12).

IX. Conclusions

1) The structure of turbulence in free shear flows is less universal than that in boundary layers. It seems to be correlated

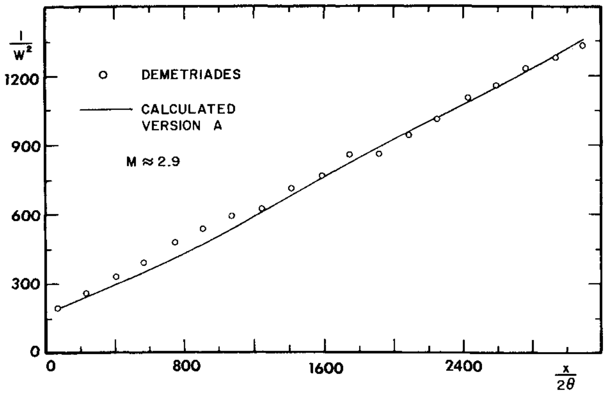

Fig. 12 Supersonic wake ($M = 2.9$).

with the spreading rate of the flow $d\delta/dx$, and the form of this dependence was deduced from comparison of predictions with experiments.

2) The primary purpose of this work was to investigate the applicability of the interaction hypothesis of Bradshaw to free shear flows. The success of the calculation procedure, developed on the basis of the hypothesis, demonstrates that it is a workable concept.

3) Due to the effect of the interaction, the empirical functions used for jets and wakes differ from those found in mixing layers. Their shapes are about the same, however, and only the magnitude of G, representing the diffusion, is changed substantially. By contrast, the interaction concept allows calculation of duct flows using the empirical functions derived from boundary-layer data without any modification.[5] Therefore, it appears that in free flows the interaction is stronger than in wall flows.

4) The results of the calculations of Version A $[L/\delta = f(d\delta/dx)]$ and Version B (using the length scale equation) are very close to each other. The only difference is that Version B consistently predicts a somewhat faster approach to the local equilibrium than Version A. An advantage of Version B is the freedom to specify initial L/δ.

5) The local spreading rate dL_0/dx of constant-pressure turbulent jets is given approximately by Eq. (23). This equation may be used for fast estimates of the spreading rate of jets needed in engineering practice.

6) The supersonic mixing layers and jets issuing into still air do not satisfy the condition $u'/U \ll 1$, used to simplify the governing equations and so the analysis is expected to be valid only up to transonic speeds. On the other hand, flows with high external velocities at the edges, do satisfy the conditions (25) and (26) up to moderate Mach numbers.

References

[1] *Proceedings of the Langley Working Conference on Free Turbulent Shear Flows,* NASA SP-321, Vol. I, July 1972.

[2] Bradshaw, P., Ferriss, D. H., and Atwell, N. P., "Calculation of Boundary Layer Development Using the Turbulent Energy Equation," *Journal of Fluid Mechanics,* Vol. 28, 1967, pp. 591–624.

[3] Lee, S. C. and Harsha, P. T., "Use of Turbulent Kinetic Energy in Free Mixing Studies," *AIAA Journal,* Vol. 8, No. 6, 1970, pp. 1026–1032.

[4] Rodi, W. and Spalding, D. B., "A Two-Parameter Model of Turbulence and its Application to Free Jets," *Wärme-und Stoffübertragung,* Vol. 3, 1970, pp. 85–95.

[5] Bradshaw, P., Dean, R. B., and McEligot, D. M., "Calculation of Interacting Turbulent Shear Layers: Duct Flow," *ASME Paper,* No. 72-WA/FE-25, Dec. 1972, New York.

[6] Townsend, A. A., *The Structure of Turbulent Shear Flow,* Cambridge University Press, New York, 1956, Chaps. V, VII.

[7] Kolmogorov, A. N., "Equations of Turbulent Motion of an Incompressible Fluid," *Izvestiya Akademii Nauk SSSR,* Ser. Fiz., VI, 1–2, 1942, pp. 56–58.

[8] Rotta, J. C., "Statistische Theorie Nichthomogener Turbulenz," *Zeitschrift für Physik,* Vol. 129, 1951, pp. 547–572.

[9] Donaldson, C. duP., "A Progress Report on an Attempt to Construct an Invariant Model of Turbulent Shear Flows," *Proceedings of AGARD Conference on Turbulent Shear Flows,* AGARD-CP-93-71, 1971, London.

[10] Chou, P. Y., "On Velocity Correlations and the Solutions of the Equations of Turbulent Fluctuations," *Quarterly Applied Mathematics,* Vol. III, No. 1, 1945, pp. 38–54.

[11] Rotta, J. C., "Recent Attempts to Develop a Generally Applicable Calculation Method for Turbulent Shear Flow Layers," *Proceedings of AGARD Conference on Turbulent Shear Flows,* AGARD-CP-93-71, 1971, London.

[12] Hanjalić, K. and Launder, B. E., "A Reynolds Stress Model of Turbulence and its Application to Thin Shear Flows," *Journal of Fluid Mecanics,* Vol. 52, 1972, pp. 609–638.

[13] Wolfstein, M., "On the Length-Scale-of-Turbulence Equation," *Israel Journal of Technology,* Vol. 8, No. 1–2, 1970, pp. 87–99.

[14] Liepmann, H. W. and Laufer, J., "Investigations of Free Turbulent Mixing," TN-1257, 1947, NACA.

[15] Bradshaw, P., Ferriss, D. H., and Johnson, R. F., "Turbulence in the Noise-Producing Region of a Circular Jet," *Journal of Fluid Mechanics,* Vol. 19, 1964, pp. 591–624.

[16] Spencer, B. W., "Statistical Investigation of Turbulent Velocity and Pressure Field in a Two-Stream Mixing Layer," Ph.D. Thesis, 1970, Nuclear Engineering, University of Illinois, Urbana, Ill.

[17] Townsend, A. A., "The Fully Developed Turbulent Wake of a Circular Cylinder," *Australian Journal of Scientific Research,* 1949, pp. 451–468.

[18] Townsend, A. A., "The Mechanism of Entrainment in Free Turbulent Flows," *Journal of Fluid Mechanics,* Vol. 26, 1966, pp. 689–715.

[19] Bradbury, L. J. S., "The Structure of a Self-Preserving Turbulent Plane Jet," *Journal of Fluid Mechanics,* Vol. 23, 1965, pp. 31–64.

[20] *Proceedings of the Langley Working Conference on Free Turbulent Shear Flows,* NASA SP-321, July 1972, Volume II, pp. 46–47.

[21] Bradbury, L. J. S., "Simple Expressions for the Spread of Turbulent Jets," *Aeronautical Quarterly,* May 1967, pp. 133–142.

[22] Chevray, R. and Kovasznay, L. S. G., "Turbulence Measurements in the Wake of a Thin Flat Plate," *AIAA Journal,* Vol. 7, No. 8, Aug. 1969, pp. 1641–1643.

[23] Gartshore, I. S., "Two-Dimensional Turbulent Wakes," *Journal of Fluid Mechanics,* Vol. 30, 1967, pp. 547–560.

[24] Bradshaw, P. and Ferriss, D. H., "Calculation of Boundary Layer Development Using the Turbulent Energy Equation: Compressible Flow on Adiabatic Walls," *Journal of Fluid Mechanics,* Vol. 46, 1971, pp. 83–110.

[25] Morel, T., "Calculation of the Free Turbulent Mixing. Interaction Approach," Ph.D. Thesis, 1972, Illinois Institute of Technology, Chicago, Ill.

[26] Morkovin, M. V., "Effects of Compressibility on Turbulent Flows," *The Mechanics of Turbulence,* edited by A. Favre, Gordon and Breach, New York, 1964, pp. 367–380.

[27] Demetriades, A., "Turbulent Mean-Flow Measurements in a Two-Dimensional Supersonic Wake," *Physics of Fluids,* Vol. 12, No. 1, 1969, pp. 24–32.